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# TWO-DIMENSIONAL SEDIMENT TRANSPORT MODELS FOR SAND-BED RIVERS

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*The problem of determining the specific mass flow of sediment entrained by a liquid flow passing above the sand bottom is studied. The boundary-value problem for a two-phase mixture of the liquid and solid particles in the active bottom layer is solved, and a general formula for determining the specific mass flow of sediment is derived. Constraints imposed on the rheological model of a moving mixture, which allow the phenomenological parameter (concentration of particles in the active layer of the mixture) to be eliminated from the model, are found. Within the framework of the proposed rheological model, the equation of riverbed deformations in the case of a sand bottom is obtained.*

**Key words:** *riverbed deformations, active bottom layer, sediment entrained by the liquid flow, equation of bottom deformations.*

**Introduction.** Based on the approach proposed in [1], the specific mass flow of solid particles $q_i$ necessary to close the two-dimensional equation of riverbed deformations is derived:

$$(1-\varepsilon)\rho_s \frac{\partial \zeta}{\partial t} + \frac{\partial q_i}{\partial s_i} = Q. \tag{1}$$

Here $\varepsilon$ is the porosity of the bottom material, $\rho_s$ is the density of bottom particles, $\zeta$ is a function that defines the shape of the bottom surface, $q_i$ is the specific mass flow of solid particles, and $Q$ is the source (sink) term. In modeling bottom deformations, the classical approach for closing Eq. (1) is the use of a dependence of the form [2–5]

$$q_i = q_0 |\tau_k^\zeta|^b \Big[\frac{\tau_i^\zeta}{|\tau_k|} - \Lambda_{ij}\,\frac{\partial \zeta}{\partial s_j}\Big] \tag{2}$$

($\tau_k^\zeta$ are the shear stresses near the bottom surface and $\tau_i^\zeta$ are the components of the stress vector on areas tangent to the surface of the mixture in the active bottom layer).

The use of Eq. (2), however, involves difficulties caused by the necessity of determining the parameters $q_0$ and $b$, the tensor $\Lambda_{ij}$, and the source term $Q$, which, in the general case, are functions depending on physicomechanical and granulometric properties of the bottom material.

The simplified formula (2) was experimentally obtained for the first time in [6] (for $q_0 \approx 8$ and $b = 1.5$). The physical grounds for this result for steady flows was given in [7, 8]. Later, formula (2) was verified in [9], where the value $b = 1$–$3$ was obtained by the probabilistic approach. By the method based on the dimension analysis, it was found [10] that $b = 1.5$–$2.5$. Developing the approach proposed in [7], the structure of the formula for the bottom with finite slopes $|\Lambda_{ij}| \neq 0$ at $b = 1.5$ was justified in [3, 5]. In these papers, however, the models of sediment entrained by the liquid flow involved two or more phenomenological coefficients, which had to be determined in particular cases through additional research.

Significant contributions to the methodology of determining the parameters $q_0$ and $b$, the tensor $\Lambda_{ij}$, and the source term $Q$ were made in [11–13]. In these papers, analytical formulas for determining the specific mass flow $q_i$

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with a power index $1.0 \leqslant b \leqslant 1.5$ were derived by means of analyzing the motion of a two-phase mixture of a liquid and solid particles with the use of the Coulomb–Newton model [11] and the Coulomb–Prandtl model [12, 13].

The formulas for calculating $q_i$ proposed in [11, 12] have one phenomenological parameters $f$ (concentration of particles in the active bottom layer), which is considered as an independent parameter and is determined in experiments. It was noted thereby that it is rather difficult to arrange such experiments.

It was demonstrated [13] that the character of the distribution of the concentration $f$ over the depth of the active layer exerts a minor effect on the specific mass flow of sediment, and the parameter $f$ can be considered as a constant with high accuracy.

In the present paper, we determine the constraint imposed on the power coefficients of the rheological model, which allows eliminating the phenomenological parameter (concentration of particles in the active layer of the mixture). It is demonstrated that the parameter $f$ (concentration of particles in the active bottom layer) used in [12, 13] is a dependent parameter, which can be eliminated in deriving formulas for the specific mass flow $q_i$, whereas the Coulomb–Newton rheological model [11] with constant viscosity of the active layer does not allow the parameter $f$ to be eliminated. Analytical dependences for $q_i$ are obtained, which make it possible to determine the quantities $\Lambda_{ij}$ and $Q$.

**1. Formulation of the Problem.** To determine the specific mass flow of solid particles, we use the formula [2–5]

$$q_i = \rho_s \int\limits_0^h f u_i \, dm, \tag{1.1}$$

where $u_i$ is the velocity vector of the active layer of the mixture and $h$ is the depth of the active layer of the mixture.

The values of the functions $u_i$, $h$, and $f$ are found by solving hydrodynamic equations obtained under the assumption that the depth of the moving active layer of sediment $h$ is small, as compared with the characteristic riverbed size in the horizontal projection, and the inertial forces are small, as compared with the friction forces ($\mathrm{Re} \ll 1$). Therefore, we can use the equations of motion [13]

$$\frac{\partial p}{\partial s_i} + \rho g \frac{\partial \zeta}{\partial s_i} + \frac{\partial \tau_i}{\partial m} = 0; \tag{1.2}$$

$$\frac{\partial p}{\partial m} = \rho g \cos\gamma, \tag{1.3}$$

supplemented by the rheological relations [14] generalized in [1]:

$$\tau_i = -(\tau_s + \tau_b) \frac{\partial u_i}{\partial m} \Big| \frac{\partial u_k}{\partial m} \Big|^{-1}. \tag{1.4}$$

Here $\tau_b = \mu_*(h - m)^n |\partial u_k/\partial m|^k$, $\tau_s = p_s \tan\varphi$ are the absolute values of the shear stresses of the liquid and solid loose phases, $n$ and $k$ are the power coefficients of the rheological model, $\tau_i$ are the components of the stress vector on areas tangent to the surface of the mixture in the active bottom layer, $\gamma$ is the acute angle between the normal to the upper boundary of the active layer [bottom surface $\zeta = \zeta(x, y)$] and the vertical line, $p$ is the pressure in the active bottom layer, $p_s = mf(\rho_s - \rho_w) g \cos\gamma$ is the pressure of solid particles suspended in water, $\rho = f\rho_s + (1 - f)\rho_w$, $\rho_w$ and $\rho_s$ are the densities of water and particles, $m$ is the coordinate on the axis directed downward, normal to the surface of the active layer [$m = 0$ on the upper boundary of the active layer $\zeta = \zeta(x, y)$], $s_i$ are the orthogonal curvilinear coordinates fitted to the bottom surface $\zeta$ (the line $s_1$ coincides with the direction of the velocity vector of the hydrodynamic flow, and the line $s_2$ is perpendicular to the latter), and $\varphi$ is the angle of internal friction of bottom particles.

According to [2, 4, 6], the criterion of the beginning of particle motion in the active layer ($m \leqslant h$) is the condition $\tau_0 < |\tau_i|$; correspondingly, the slip surface at the depth $h$ determining the lower boundary of the active layer is found from the condition

$$|\tau_i(h)| = \tau_0, \tag{1.5}$$

where $\tau_0 = p_s \tan\varphi$ is the shear stress at which bottom particles start moving [4, 6]. By virtue of continuity of the velocities $u_i$ and stresses $|\tau_i|$ on the slip surface, we obtain the following boundary conditions from Eq. (1.5):

$$u_i(h) = 0. \tag{1.6}$$

Conditions (1.5), (1.6) and conditions for the normal and shear stresses imposed on the upper boundary of the active bottom layer ($m = 0$)

$$\tau_i(0) = \tau_i^\zeta; \tag{1.7}$$

$$p(0) = p^\zeta \tag{1.8}$$

($p^\zeta$ is the normal pressure on the upper boundary of the active layer) allow us to close problem (1.2), (1.4) and obtain an analytical expression for the specific mass flow.

**2. Determining the Specific Mass Flow of Sediment.** From Eq. (1.3) and condition (1.8), we obtain the expression for the function of the pressure $p$ in the active layer:

$$p = p^\zeta + m\rho g \cos\gamma.$$

Here $p^\zeta = \rho_w g(H - \zeta)$. Assuming that the slope of the free surface of the river flow is much smaller than the slope of the bottom $|\partial H/\partial s_i| \ll |\partial\zeta/\partial s_i|$ ($H$ is the free surface of the river flow), we obtain

$$\frac{\partial p}{\partial s_i} = -\rho_w g\,\frac{\partial\zeta}{\partial s_i}, \qquad \frac{\partial p}{\partial s_i} + (f\rho_s + (1-f)\rho_w)g\,\frac{\partial\zeta}{\partial s_i} = f(\rho_s - \rho_w)g\,\frac{\partial\zeta}{\partial s_i} = A\Gamma_i, \tag{2.1}$$

where

$$\Gamma_i = \frac{1}{\tan\varphi\cos\gamma}\,\frac{\partial\zeta}{\partial s_i}; \tag{2.2}$$

$$A = F_a f, \qquad F_a = \cos\gamma\tan\varphi\,(\rho_s - \rho_w)g. \tag{2.3}$$

Integrating Eqs. (1.2) over the depth of the active layer and taking into account Eqs. (2.1)–(2.3), we obtain

$$\tau_i = \tau_i^\zeta - mA\Gamma_i. \tag{2.4}$$

Let us match the axis $s_1$ with the vector $\boldsymbol{\tau}^\xi$; then the vector $\boldsymbol{\tau}^\zeta$ in the coordinate system $s_i$ has the components $\tau_i^\zeta = (\tau_1^\zeta, 0)$.

Using Eqs. (1.4) and (2.4) and the boundary conditions (1.5) and (1.6) and taking into account that $\tau_b \equiv 0$ and $\tau_s = hA$ for $m = h$, we find the depth of the active layer $h$:

$$\sqrt{(\tau_1^\zeta - hA\Gamma_1)^2 + (hA\Gamma_2)^2} = hA. \tag{2.5}$$

Choosing only the positive root $h$ inside a circle of unit radius $|\Gamma_i| < 1$, we obtain

$$h = \frac{\tau_1^\zeta}{A\sqrt{1-\Gamma_2^2} + \Gamma_1}.$$

To determine the velocity gradient $\partial u_i/\partial m$ necessary for calculating the specific mass flow of solid particles (1.1), we use the relation

$$\frac{\partial u_i}{\partial m} = -\frac{\tau_i}{|\tau_k|}\,\Big|\frac{\partial u_k}{\partial m}\Big|. \tag{2.6}$$

We find the absolute value $|\partial u_i/\partial m|$ in Eq. (2.6) from the equation of state (1.4):

$$\Big|\frac{\partial u}{\partial m}\Big| = \sqrt[k]{\frac{|\tau_i| - mA}{\mu_*(h-m)^n}}. \tag{2.7}$$

Eliminating the absolute value of stresses at each point of the active layer $|\tau_i| = |\tau_1^\zeta - mA\Gamma_i| = A\sqrt{(\tau_1^\zeta - m\Gamma_1)^2 + (m\Gamma_2)^2}$ from Eq. (2.7) and expressing the stresses on the layer surface $\tau_1^\zeta = hA(\sqrt{1-\Gamma_2^2} + \Gamma_1)$ via the depth $h$, we find

$$\Big|\frac{\partial u_k}{\partial m}\Big| = \Big(\frac{A}{\mu_*}\Big)^{1/k} \frac{1}{(h-m)^{n/k}} \sqrt[k]{\sqrt{(h(\sqrt{1-\Gamma_2^2}+\Gamma_1)-m\Gamma_1)^2+(m\Gamma_2)^2}-m}. \tag{2.8}$$

Integrating Eq. (1.1) by parts, we obtain

$$q_i = \rho_s \int_0^h f u_i\, dm = \rho_s f u_i m \Big|_0^h - \rho_s f \int_0^h m \frac{\partial u_i}{\partial m}\, dm = \rho_s f \int_0^h m \Big|\frac{\partial u_k}{\partial m}\Big| \frac{\tau_i}{|\tau_k|}\, dm. \tag{2.9}$$

By virtue of the boundary conditions (1.6), the first term in the right side of Eq. (2.9) equals zero. To find the second term, we substitute Eq. (2.8) into Eq. (2.9) and expand the integrands in Eq. (2.9) with respect to the parameter $\Gamma_i$ with accuracy to the second order:

$$q_1 = \Big(\frac{A}{\mu_*}\Big)^{1/k} \int_0^h \frac{f\rho_s m}{(h-m)^{n/k}} \sqrt[k]{\sqrt{(h(\sqrt{1-\Gamma_2^2}+\Gamma_1)-m\Gamma_1)^2+(m\Gamma_2)^2}-m}$$

$$\times \frac{h(\sqrt{1-\Gamma_2^2}+\Gamma_1)-m\Gamma_1}{\sqrt{(h(\sqrt{1-\Gamma_2^2}+\Gamma_1)-m\Gamma_1)^2+(m\Gamma_2)^2}}\, dm \approx \Big(\frac{F_a}{\mu_*}\Big)^{1/k} \frac{\rho_s f^{(k+1)/k} k^2 h^{(2k-n+1)/k}}{(n-1-k)(n-1-2k)}$$

$$\times \Big(1+\frac{\Gamma_1}{k} - \frac{1+k}{2k^2}\,\Gamma_1^2 - \frac{(n-1)^2+k(6k^2+20k+9)-9kn}{2k(n-1-3k)(n-1-4k)}\,\Gamma_2^2\Big); \tag{2.10}$$

$$q_2 = \Big(\frac{F_a}{\mu_*}\Big)^{1/k} \frac{\rho_s f^{(k+1)/k} h^{(2k-n+1)/k}}{(n-1-k)(n-1-2k)} \frac{k^3}{n-1-3k}\,\Gamma_2. \tag{2.11}$$

Eliminating the depth of the active layer $h$ from Eqs. (2.10) and (2.11), we obtain

$$q_1 = \Big(\frac{F_a}{\mu_*}\Big)^{1/k} k^2 \rho_s f^{(k+1)/k} \frac{(\tau_1^\zeta/(F_a f\sqrt{1-\Gamma_2^2}+\Gamma_1))^{(2k-n+1)/k}}{(n-1-k)(n-1-2k)}$$

$$\times \Big(1+\frac{\Gamma_1}{k} - \frac{1+k}{2k^2}\,\Gamma_1^2 - \frac{(n-1)^2+k(6k^2+20k+9)-9kn}{2k(n-1-3k)(n-1-4k)}\,\Gamma_2^2\Big); \tag{2.12}$$

$$q_2 = \Big(\frac{F_a}{\mu_*}\Big)^{1/k} \frac{\rho_s f^{(k+1)/k}(\tau_1^\zeta/(F_a f\sqrt{1-\Gamma_2^2}+\Gamma_1))^{(2k-n+1)/k}}{(n-1-k)(n-1-2k)} \frac{k^3}{n-1-3k}\,\Gamma_2. \tag{2.13}$$

As in the one-dimensional case [1], it follows from Eqs. (2.12) and (2.13) that the phenomenological parameter $f$ is eliminated from the process of calculating the specific mas flow $q_i$ if the power coefficients are identical $n \equiv k$. In this case, Eqs. (2.12) and (2.13) acquire the form

$$q_1 = \Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{\rho_s k^2 \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{(k+1)F_a(\sqrt{1-\Gamma_2^2}+\Gamma_1)^{(k+1)/k}} \Big(1+\frac{\Gamma_1}{k}+\frac{k-1}{2k^2}\,\Gamma_1^2 - \frac{(k+1)(6k^2+6k+1)}{2k(2k+1)(3k+1)}\,\Gamma_2^2\Big); \tag{2.14}$$

$$q_2 = -\Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{\rho_s k^3 \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{(k+1)(2k+1)F_a} \frac{\Gamma_2}{(\sqrt{1-\Gamma_2^2}+\Gamma_1)^{(k+1)/k}}. \tag{2.15}$$

The condition $n \equiv k$ reflects the constraints imposed on the choice of the equation of state in formulating problem (1.1)–(1.8). As dependence (2.4) obtained from the equilibrium equations requires the shear stress in the active layer to decrease with increasing depth of the layer, the equation of state (1.4) used should be capable of reproducing this behavior of stresses in the active layer [12, 13], which does not happen, for instance, in the case of the Coulomb–Newton medium [11] ($n = 0$, $k = 1$) where the stress increases with increasing $m$.

To eliminate the quantities $\Gamma_1$ and $\Gamma_2$ from the denominator of Eqs. (2.14) and (2.15), we again expand these expressions with respect to these parameters with second-order accuracy:

$$q_1 = \Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{\rho_s k^2 \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{(k+1)F_a(\sqrt{1-\Gamma_2^2}+\Gamma_1)^{(k+1)/k}} \times \Big(1 + \frac{\Gamma_1}{k} + \frac{k-1}{2k^2}\,\Gamma_1^2 - \frac{(k+1)(6k^2+6k+1)}{2k(2k+1)(3k+1)}\Gamma_2^2\Big)$$

$$\approx \Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{k^2}{k+1}\,\frac{\rho_s \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{F_a}\,\Big(1 - \Gamma_1 + \Gamma_1^2 - \frac{k+1}{2(2k+1)(3k+1)}\,\Gamma_2^2\Big); \qquad (2.16)$$

$$q_2 = -\Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{\rho_s k^3 \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{(k+1)(2k+1)F_a}\,\frac{\Gamma_2}{(\sqrt{1-\Gamma_2^2}+\Gamma_1)^{(k+1)/k}}$$

$$\approx -\Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{2k^3}{(k+1)(2k+1)(3k+1)}\,\frac{\rho_s \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{F_a}\,((1+3k)\Gamma_2 - (2+3k)\Gamma_1\Gamma_2). \qquad (2.17)$$

For comparisons with Eq. (2) and formulas derived in [11, 12], we reject terms of the second order with respect to $\Gamma_i$ in Eqs. (2.16), (2.17) and write the resultant equations in the vector form:

$$q_i = \Big(\frac{1}{\mu_*}\Big)^{1/k} \frac{k^2}{k+1}\,\frac{\rho_s \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{F_a}\Big(\Big(1 - \frac{\Gamma_1}{2k+1}\Big)\frac{\tau_i^\zeta}{|\tau_k^\zeta|} - \frac{2k}{2k+1}\,\Gamma_i\Big). \qquad (2.18)$$

We can easily demonstrate that Eq. (2.18) is similar to the model formula (2); in contrast of the latter, however, Eq. (2.18) does not contain any phenomenological coefficients, except for the rheological power parameter $k$, which is usually assumed to be equal to 2.

A particular case of Eq. (2.18) is formulas obtained in [12, 13] (for the parameters $k = 2$ and $\mu_* = \rho_w \varkappa^2$, where $\varkappa$ is the Kármán constant):

$$q_i = \frac{4}{3}\,\frac{\rho_s (\tau_1^\zeta)^{3/2}}{\varkappa\sqrt{\rho_w}\,F_a}\Big[\Big(1 - \frac{1}{5}\,\Gamma_1\Big)\frac{\tau_i^\zeta}{|\tau_k^\zeta|} - \frac{4}{5}\,\Gamma_i\Big].$$

**3. Allowance for the Relative Velocity of Particles.** If Eq. (1.1) is used to determine the specific mass flow of solid particles (one-velocity model), the flow rates of the solid particles and the liquid are assumed to be identical. For this reason, the value of velocity at the beginning of particle motion cannot be determined. To use a more precise two-velocity formula, it is necessary to find the velocity $v_i$ of delay of the particles behind the hydrodynamic flow and then to perform a more precise calculation of the specific mass flow of the solid particles:

$$q_i = \rho_s f \int_0^h (u_i - v_i)\, dm. \qquad (3.1)$$

In practice, it is sufficient to find a correction to the specific mass flow (2.18), which has the form

$$\Delta q_i = -\rho_s f \int_0^h v_i\, dm. \qquad (3.2)$$

Following [12, 13], we find the velocity of particle delay $v_i$, which is necessary to calculate correction (3.2), by analyzing the balance of forces acting on the particles in a unit volume:

$$F_i^w + F_i^s + F_i^g = 0. \qquad (3.3)$$

Here $F_i^w$ is the drag force acting on the particles from the side of water [3, 12]:

$$F_i^w = f\,\frac{c_x \rho_w v_i |v_k|}{2d}, \qquad (3.4)$$

$v_i$ are the components of the relative velocity of particle motion, $d$ is the particle diameter, $c_x$ is the drag coefficient of the particles, $F_i^s$ is the force of friction between the particles, which is determined from the rheological relation (1.4) for the stress of the solid phase:

$$F_i^s = -fF_a\,\frac{\partial m e_i}{\partial m}, \qquad (3.5)$$

$e_i$ is the unit vector of the direction of the flow of entrained particles, which depends on the depth of the active layer [13]:

$$e_1 = 1 - \frac{1}{2}\Big(\frac{m}{h}\,\Gamma_2\Big)^2, \qquad e_2 = -\frac{m}{h}\,\Gamma_2\Big(1 - \Big(1 - \frac{m}{h}\Big)\Gamma_1\Big),$$

and $F_i^g$ is the gravity force in the projection onto the plane tangent to the bottom surface:

$$F_i^g = -fF_a\Gamma_i. \qquad (3.6)$$

From Eqs. (3.3)–(3.6), we obtain the expression for the relative velocity $v_i$

$$\frac{c_x \rho_w v_i |v_k|}{2d} = F_a(1 - \Gamma_i)E_i, \tag{3.7}$$

where

$$E_i = \Big(1 + \Gamma_1 - \frac{2m^2\Gamma_2^2}{h^2},\ \Gamma_2 - \frac{m^2}{h^2}\,\Gamma_1\Gamma_2 - \frac{2m\Gamma_2(h - (h-m)\Gamma_1)}{h^2}\Big).$$

From Eq. (3.7), we find

$$v_i = v^* \frac{E_i}{\sqrt{|E_k|}},$$

where $v^* = \sqrt{2dF_a/(c_x\rho_w)}$.

Expanding the term $E_i/\sqrt{E_i}$ with respect to the powers of $\Gamma_i$ and retaining terms of the first order of smallness, we obtain

$$\frac{E_1}{\sqrt{E_k}} \approx 1 + \frac{\Gamma_1}{2}; \tag{3.8}$$

$$\frac{E_2}{\sqrt{E_k}} \approx \frac{h - 2m}{h}\,\Gamma_2. \tag{3.9}$$

Using Eqs. (3.8) and (3.9), we find the correction for the mass flow (3.2):

$$\Delta q_1 = \rho_s f v^* \int\limits_0^h v_1\,dm = \rho_s f v^* \Big(1 + \frac{1}{2}\,\Gamma_1\Big)h = \rho_s v^* \Big(1 - \frac{1}{2}\,\Gamma_1\Big)\frac{\tau_1^\zeta}{F_a}; \tag{3.10}$$

$$\Delta q_2 = \rho_s f v^* \int\limits_0^h v_2\,dm = 0. \tag{3.11}$$

The solution of Eqs. (3.10) and (3.11) shows that the correction for the mass flow of the solid particles in the direction $s_2$ is equal to zero, because the velocity of the hydrodynamic flow in this direction equals zero; a certain mass flow of sediment in the direction $s_2$ can exist, however, because of the bottom slope. Hence, in determining the criterion of the beginning of motion of solid particles in model (3.1), it is necessary to use only velocity components in the direction $s_1$, i.e., $|v_1| < |u_1|$.

Formulas (3.10) and (3.11) can be presented in the form

$$\Delta q_i = \rho_s v^* \Big(1 - \frac{1}{2}\,\Gamma_i\Big)\frac{\tau_i^\zeta}{F_a}. \tag{3.12}$$

With allowance for correction (3.12), the expression for the specific mass flow of solid particles takes the form

$$q_i = \frac{k^2}{\sqrt[k]{\mu_*}\,(k+1)}\,\frac{\rho_s \tau_1^\zeta (\tau_1^\zeta)^{1/k}}{F_a}\,\Big(\Big(1 - \frac{\Gamma_1}{2k+1}\Big)\frac{\tau_i^\zeta}{|\tau_k^\zeta|} - \frac{2k}{2k+1}\,\Gamma_i\Big) - \rho_s \sqrt{\frac{2d}{c_x\rho_w F_a}}\,\Big(1 - \frac{1}{2}\,\Gamma_i\Big)\tau_1^\zeta\,\frac{\tau_i^\zeta}{|\tau_k^\zeta|}. \tag{3.13}$$

From Eq. (3.13), we can obtain the condition of the beginning of sediment motion:

$$1 - \sqrt{\frac{(k+1)^2}{k^4}\,\frac{2dF_a}{c_x\rho_w}\Big(\frac{\mu_*}{\tau_1^\zeta}\Big)^{2/k}} - \Big(1 - \sqrt{\frac{(k+1)^2}{k^4}\,\frac{dF_a}{2\rho_w c_x}\Big(\frac{\mu_*}{\tau_1^\zeta}\Big)^{2/k}}\Big)\Gamma_1 = 0,$$

which allows us to determine the critical stresses $\tau_*$:

$$\tau_* = \mu_* \Big(\frac{dF_a(2-\Gamma_1)^2(k+1)^2}{2k^4 c_x\rho_w(1-\Gamma_1)^2}\Big)^{k/2}. \tag{3.14}$$

For $k = 2$ and $\mu_* = \varkappa^2\rho_w$, Eq. (3.14) coincides with the formulas in [12, 13].

**4. Equation for Riverbed Deformations.** Substituting the expression for the specific mass flow of solid particles (3.13) into Eq. (1) and taking into account Eq. (2.2), we obtain the equation for riverbed deformations in the final form

$$(1-\varepsilon)\,\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial s_i}\Big(\Lambda_{ij}\,\frac{\partial \zeta}{\partial s_j}\Big) + Q,$$

where

$$\Lambda_{ij} = \frac{mk^2(\tau_1^\zeta)^{(k+1)/k}}{\sqrt[k]{\mu_*}\,(k+1)F_a \tan\varphi \cos\gamma} \begin{bmatrix} 1 - \dfrac{k+1}{k^2}\sqrt[k]{\dfrac{\mu_*}{\tau_1^\zeta}}\sqrt{\dfrac{dF_a}{2c_x\rho_w}} & 0 \\ 0 & \dfrac{2k}{2k+1} \end{bmatrix},$$

$$Q = m\,\frac{\partial}{\partial s_i}\Big[\frac{\tau_1^\zeta}{F_a}\Big(\frac{k^2}{k+1}\sqrt[k]{\frac{\tau_1^\zeta}{\mu_*}} - \sqrt{\frac{2dF_a}{c_x\rho_w}}\Big)\frac{\tau_i^\zeta}{|\tau_k^\zeta|}\Big], \qquad m = \begin{cases} 1, & \tau_1^\zeta > \tau_*, \\ 0, & \tau_1^\zeta \leqslant \tau_*. \end{cases}$$

**5. Conclusions.** The study performed allows the following conclusions to be drawn.

As a result of solving the boundary-value problem for a two-phase mixture of a liquid and solid particles in the active bottom layer, a general formula (3.13) was derived for determining the specific mass flow of sediment; this formula contains no phenomenological parameters. The formula generalizes the known models [12, 13] and agrees well with experimental data for the horizontal bottom [2–4] and for the bottom with finite transverse [14] and longitudinal [15] slopes.

Constraints imposed on the rheological model of a moving mixture were determined, which allow the only phenomenological parameter $f$ (concentration of particles in the active layer of the mixture) to be eliminated from the model.

It was demonstrated that the parameter $f$ cannot be eliminated if the Coulomb–Newton model [11] is used.

Analytical dependences for determining the parameters $q_0$, $b$, $\Lambda_{ij}$, and $Q$ were derived.

The equation of riverbed deformations for a sand bottom was obtained within the framework of the rheological model proposed.

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